

Computation of Delta sets of numerical monoids

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Abstract

Let $\{a_1, \dots, a_p\}$ be the minimal generating set of a numerical monoid S . For any $s \in S$, its Delta set is defined by $\Delta(s) = \{l_i - l_{i-1} \mid i = 2, \dots, k\}$ where $\{l_1 < \dots < l_k\}$ is the set $\{\sum_{i=1}^p x_i \mid s = \sum_{i=1}^p x_i a_i \text{ and } x_i \in \mathbb{N} \text{ for all } i\}$. The Delta set of a numerical monoid S , denoted by $\Delta(S)$, is the union of all the sets $\Delta(s)$ with $s \in S$. As proved in [5], there exists a bound N such that $\Delta(S)$ is the union of the sets $\Delta(s)$ with $s \in S$ and $s < N$. In this work, we obtain a sharpened bound and we present an algorithm for the computation of $\Delta(S)$ that requires only the factorizations of a_1 elements.

Keywords: Delta set, non-unique factorization, numerical monoid, numerical semigroup.

MSC-class: 20M14 (Primary), 20M05 (Secondary).

Introduction

The study of the structure of $\Delta(S)$ and its computation plays an important role in the theory of non-unique factorization. For example in [2], it found a rigorous study of $\Delta(S)$ for numerical monoids that shows the structure of $\Delta(S)$ can be very complex even in the case S is generated by only three elements. Also in [2], some bounds for the maximum and the minimum of $\Delta(S)$ with S a numerical monoid are given. Another interesting work is [4] where some results concerning the structure of the Delta sets of BF-monoids are proved and it is shown that the minimum and the maximum of $\Delta(S)$ can be completely determined using the Betti elements of S . In [7], the conditions which must be satisfied by the generators of $S = \langle a_1, a_2, a_3 \rangle$ for $\Delta(S)$ being a singleton are shown. One of the main results used to compute $\Delta(S)$ is given in [10]; it proves that every commutative cancellative reduced atomic monoid S satisfies that $\min(\Delta(S)) = \gcd(\Delta(S))$. A method for computing $\Delta(S)$ is found in [5]; in

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that paper, it is proved that for every numerical monoid S with minimal system of generators $a_1 < \dots < a_p$, and for every element $s \in S$ such that $s \geq 2pa_2a_p^2$ it must hold $\Delta(s) = \Delta(s + a_1a_p)$. Thus, for a primitive numerical monoid S we have $\Delta(S) = \cup_{s \in S, s < N} \Delta(s)$ with $N = 2pa_2a_p^2 + a_1a_p$, and this implies that the computation of $\Delta(S)$ requires only a finite number of steps.

A different approach is to study $\Delta(S)$ in different types of monoids. For instance in [2], it is proved that if a_1 and a_2 are integers satisfying $1 < a_1 < a_2$ and $\gcd(a_1, a_2) = 1$, then $\Delta(\langle a_1, a_2 \rangle) = \{a_2 - a_1\}$. If S is a numerical monoid generated by a k -interval, then $\Delta(S) = \{k\}$, and if $S = \langle n, n + k, (k + 1)n - k \rangle$ where $n \geq 3$, $k \geq 1$ and $\gcd(n, k) = 1$, then $\Delta(S) = [k, 2k, \dots, \lfloor \frac{n+k-1}{k+2} \rfloor k]$. In [7], the elements of the set $\Delta(S)$ with $S = \langle a_1, a_2, a_3 \rangle$ a numerical monoid are characterized, and in the case $a_1 = 3$ it is proved that $\{\frac{a_2+a_3}{3} - 2\} \subseteq \Delta(S) \subseteq [1, \frac{a_2+a_3}{3} - 2] \cap \mathbb{N}$. In [6], it is shown that for an increasing sequence r_1, \dots, r_t of positive integers, a positive integer n and $S_n = \langle n, n + r_1, \dots, n + r_t \rangle$ a numerical monoid, there exists a positive integer N such that if $n > N$, then $|\Delta(S_n)| = 1$. Other works in this area may be found in [1, 3, 11, 12, 13].

Despite the amount of existing works, the computation of $\Delta(S)$ for a given numerical monoid is not an easy task. The main problems are the high values of the bounds and the large amount of factorizations that are required even in the cases the bound is low. In this work we cover some gaps in the knowledge of the Delta sets of numerical monoids. We give explicit bounds that improve the bounds obtained in previous works and we also use some improvements in the computation of the expressions of some elements. All these advances allow us to get a better algorithm to compute the Delta sets of numerical monoids. The theoretical results of this work are complemented with the software [9] developed in `Mathematica` that provides us functions to compute the Delta set of a numerical monoid.

The contents of this paper are organized as follows. In Section 1, we introduce some definitions and notations used in this work. In Section 2, we study the structure of $\Delta(S)$. These results are used to get the existence of the bound N_S . In Section 3, a formulation of N_S is given. Finally, in Section 4, we give an algorithm to compute $\Delta(S)$, and we illustrate our method with some examples showing their execution times.

1 Preliminaries

Let \mathbb{N} be the set of nonnegative integers and let \mathbb{Q}_{\geq} be the set of nonnegative rational numbers. If S is an additive submonoid of \mathbb{N} , then S is called a numerical monoid. We say that the integers a_1, \dots, a_p with $p \in \mathbb{N} \setminus \{0\}$ generate S if $S = \{x_1a_1 + \dots + x_pa_p \mid x_i \in \mathbb{N} \text{ for all } i = 1, \dots, p\}$; this is denoted by $S = \langle a_1, \dots, a_p \rangle$. It is well known that the minimal (in terms of cardinality and set inclusion) generating set of S is unique. In the sequel, we assume that $\{a_1, \dots, a_p\}$ is the minimal generating set of S and $a_1 < \dots < a_p$. A numerical monoid $S = \langle a_1, \dots, a_p \rangle$ is primitive when $\gcd(a_1, \dots, a_p) = 1$; these monoids are also known as numerical semigroups and every numerical monoid is isomorphic to a primitive numerical monoid. Hence, we can narrow our study to the primitive case. If S is a primitive numerical monoid, then there exists an integer $\mathcal{F}(S) \notin S$ such that $s > \mathcal{F}(S)$ implies that $s \in S$. This integer is known as the Frobenius number of S . For more details on numerical monoids, the reader is

directed to the monograph [15].

For numerical monoid $S = \langle a_1, \dots, a_p \rangle$, it must hold $S \cong \mathbb{N}^p / \sim_M$, where M is the subgroup of \mathbb{Z}^p of rank $p-1$ defined by the equation $a_1x_1 + \dots + a_px_p = 0$ and \sim_M is defined as $x \sim_M y$ if and only if $x - y \in M$ for all $x, y \in \mathbb{N}^p$ (see [14] for further details). Denote by $Z(s)$ the set $\{(x_1, \dots, x_p) \in \mathbb{N}^p \mid \sum_{i=1}^p x_i a_i = s\}$ for every $s \in \mathbb{N}$. For all $x, y \in \mathbb{N}^p$ and every $s \in S$, two elements x, y belong to $Z(s)$ if and only if $x \sim_M y$. Define the linear function $L : \mathbb{Q}^p \rightarrow \mathbb{Q}$ with $L(x_1, \dots, x_p) = \sum_{i=1}^p x_i a_i$.

Definition 1. Given $s \in S$ and $S = \langle a_1, \dots, a_p \rangle$, set $\mathcal{L}(s) = \{L(x_1, \dots, x_p) \mid (x_1, \dots, x_p) \in Z(s)\}$, which is known as the set of lengths of s in S . Since S is a numerical monoid, it is not hard to prove that this set of lengths is bounded, and so there exist some positive integers $l_1 < \dots < l_k$ such that $\mathcal{L}(s) = \{l_1, \dots, l_k\}$. The set

$$\Delta(s) = \{l_i - l_{i-1} : 2 \leq i \leq k\}$$

is known as the Delta set of s . We globalize the notion of the Delta set by setting

$$\Delta(S) = \bigcup_{s \in S} \Delta(s).$$

The set $\Delta(S)$ is called the Delta set of S .

2 The structure of $\Delta(S)$

The computation of $\Delta(S)$ with S a numerical monoid generated by two elements is solved in [2]. Hence, we only consider primitive numerical monoids minimally generated by at least three elements. Denote by $\{e_1, \dots, e_p\}$ the canonical basis of \mathbb{R}^p .

Lemma 2. Let $S = \langle a_1, \dots, a_p \rangle \cong \mathbb{N}^p / \sim_M$ be a numerical monoid. Then $\min(\Delta(S)) = \min\{L(m) \mid L(m) > 0, m \in M\}$. Furthermore, if $M = \langle m_1, \dots, m_{p-1} \rangle$, then $\min(\Delta(S)) = \gcd(L(m_1), \dots, L(m_{p-1}))$.

Proof. Since $a_2e_1 - a_1e_2 \in M$ and $L(a_2e_1 - a_1e_2) = a_2 - a_1 > 0$, then $\{L(m) \mid L(m) > 0, m \in M\} \neq \emptyset$.

For every $l \in \Delta(S)$, there exist $s \in S$ and $\gamma, \gamma' \in Z(s)$ such that $l = L(\gamma) - L(\gamma') = L(\gamma - \gamma')$. Since $\gamma - \gamma' \in M$, we obtain that $l \geq \min\{L(m) \mid L(m) > 0, m \in M\}$, and thus $\min(\Delta(S)) \geq \min\{L(m) \mid L(m) > 0, m \in M\}$.

Let m be an element of M such that $L(m) > 0$. It is easy to find $\gamma \in \mathbb{N}^p$ fulfilling that $\gamma + m \in \mathbb{N}^p$. Clearly, there exists $s \in S$ such that $\gamma, \gamma + m \in Z(s)$. Since $L(m) > 0$, using the linearity of L , we have $L(\gamma) < L(\gamma + m)$. If $\mathcal{L}(s) = \{l_1, \dots, l_t\}$, there exist $1 \leq i < j \leq t$ such that $l_i = L(\gamma)$ and $l_j = L(\gamma + m)$. We distinguish two cases: $i + 1 = j$ and $i + 1 < j$. If $i + 1 = j$, then $L(m) = L(\gamma + m) - L(\gamma) = l_{i+1} - l_i \in \Delta(s)$, and therefore $\min(\Delta(S)) \leq \min(\Delta(s)) \leq L(m)$. If $i + 1 < j$, then $L(m) = l_j - l_i > l_{i+1} - l_i \in \Delta(s)$, and thus $\min(\Delta(S)) \leq \min(\Delta(s)) < L(m)$. Thus, $\min(\Delta(S)) \leq \min\{L(m) \mid m \in M, L(m) > 0\}$.

We have proved that $\min(\Delta(S)) = \min\{L(m) \mid m \in M, L(m) > 0\}$. Since $\{m_1, \dots, m_{p-1}\}$ is a system of generators of M , from the linearity of L we deduce that $\min\{L(m) \mid m \in M, L(m) > 0\} = \gcd(L(m_1), \dots, L(m_{p-1}))$. This implies that $\min(\Delta(S)) = \gcd(L(m_1), \dots, L(m_{p-1}))$. \square

On the sequel we denote by d the element $\min(\Delta(S))$. One of the consequences of Lemma 2 is that d divides $L(\gamma) - L(\gamma')$ for every $s \in S$ and every $\gamma, \gamma' \in Z(s)$, and therefore d divides all the elements of $\Delta(S)$.

Definition 3. Under the assumptions of Lemma 2, there exist $u_1, \dots, u_{p-1} \in \mathbb{Z}$ such that $d = u_1 L(m_1) + \dots + u_{p-1} L(m_{p-1})$. We refer to the vectors $\vec{v} = u_1 m_1 + \dots + u_{p-1} m_{p-1} \in M$ as the minimum length increase vectors.

Definition 4. Under the assumptions of Lemma 2, define $\vec{h} = \frac{d}{a_p - a_1} (a_p e_1 - a_1 e_p) \in \mathbb{Q}^p$.

Note that every minimum length increase vector \vec{v} and the vector \vec{h} verify $L(\vec{v}) = L(\vec{h}) = d$.

Consider E the subgroup of \mathbb{Z}^p defined by the equation $x_1 + \dots + x_p = 0$ (note that for every $\vec{c} \in E$, $L(\vec{c}) = 0$). Denote by $V(E)$ the vectorial subspace of \mathbb{Q}^p defined by the equation $x_1 + \dots + x_p = 0$. It satisfies $E \subset V(E)$, $\dim(V(E)) = p - 1$ and $\vec{v}, \vec{h} \notin V(E)$. For every $s \in \mathbb{N}$, let \mathcal{H}_s be the affine hyperplane $\{(x_1, \dots, x_p) \in \mathbb{Q}^p \mid \sum_{i=1}^p x_i a_i = s\}$. The set \mathcal{H}_0 is the vectorial subspace over \mathbb{Q} generated by M , its defining equation is $a_1 x_1 + \dots + a_p x_p = 0$ and it verifies $\dim(\mathcal{H}_0) = p - 1$.

Definition 5. Let $S = \langle a_1, \dots, a_p \rangle$ be a numerical monoid. Define $\vec{q}_i = \frac{1}{\gcd(a_i - a_p, -a_1 + a_p, a_1 - a_i)} ((a_i - a_p)e_i + (-a_1 + a_p)e_i + (a_1 - a_i)e_p) \in \mathbb{Z}^p$ for every $i = 2, \dots, p - 1$.

It is straightforward to prove that the vectors \vec{q}_i verify the defining equations of M and E . Therefore, $\vec{q}_i \in M \cap E$ for every $i = 2, \dots, p - 1$. Note that, since the i th coordinate of \vec{q}_i is greater than zero for all $i = 2, \dots, p - 2$ (recall that $a_1 < \dots < a_p$), the set $\{\vec{q}_2, \dots, \vec{q}_{p-1}\}$ is \mathbb{Q} -linearly independent.

Remark 6. The defining equations of $V(E)$ and \mathcal{H}_0 are \mathbb{Q} -linearly independent. Thus, $\dim(\mathcal{H}_0 \cap V(E)) = p - 2$. Since $\{\vec{q}_2, \dots, \vec{q}_{p-1}\} \subset M \cap E \subset \mathcal{H}_0 \cap V(E)$ is a linearly independent set, this set is a basis of $\mathcal{H}_0 \cap V(E)$.

Definition 7. Let $S = \langle a_1, \dots, a_p \rangle$ be a numerical monoid. For all $s \in \mathbb{N}$ and for all $i \in \{1, \dots, p\}$, define $X_i(s) = \frac{s}{a_i} e_i \in \mathbb{Q}_{\geq}^p$, the point of intersection of the affine hyperplane \mathcal{H}_s with the x_i -axis. These elements verify $L(X_i(s)) = s/a_i$ and $L(X_1(s)) > \dots > L(X_p(s))$.

Definition 8. Let $S = \langle a_1, \dots, a_p \rangle$ be a numerical monoid. For all $s \in \mathbb{N}$ and for all $i \in \{1, \dots, p\}$, define $P_i(s) = \frac{s(a_i - a_p)}{a_i(a_1 - a_p)} e_1 + \frac{s(a_1 - a_i)}{a_i(a_1 - a_p)} e_p \in \mathbb{Q}_{\geq}^p$. Clearly, $P_1(s) = X_1(s)$, $P_p(s) = X_p(s)$, $L(P_i(s)) = L(X_i(s))$ for all $i \in \{2, \dots, p - 1\}$ and $L(X_1(s)) = L(P_1(s)) > L(P_2(s)) > \dots > L(P_{p-1}(s)) > L(P_p(s)) = L(X_p(s))$.

For every $A, B \in \mathbb{Q}^p$, denote by \overline{AB} the set $\{A + \lambda \overrightarrow{AB} \mid \lambda \in \mathbb{Q}, 0 \leq \lambda \leq 1\}$, the line segment with endpoints A and B . Note that the points $P_i(s)$ can be expressed as $P_i(s) = \frac{a_1(a_i - a_p)}{a_i(a_1 - a_p)} X_1(s) + \frac{a_p(a_1 - a_i)}{a_i(a_1 - a_p)} X_p(s)$ where $\frac{a_1(a_i - a_p)}{a_i(a_1 - a_p)} + \frac{a_p(a_1 - a_i)}{a_i(a_1 - a_p)} = 1$, $0 \leq \frac{a_1(a_i - a_p)}{a_i(a_1 - a_p)}$ and $0 \leq \frac{a_p(a_1 - a_i)}{a_i(a_1 - a_p)}$. This implies that $P_i(s) = X_1(s) + \left(1 - \frac{a_1(a_i - a_p)}{a_i(a_1 - a_p)}\right) \overrightarrow{X_1(s)X_p(s)}$, and thus $P_i(s) \in \overline{X_1(s)X_p(s)}$.

Denote by r the line defined by $X_1(s)$ and $X_p(s)$. The point $X_1(s)$ belongs to the x_1 -axis and $X_p(s)$ to the x_p -axis. This implies that $\overline{X_1(s)X_p(s)}$ is equal

to $r \cap \mathbb{Q}_{\geq}^p$. For every $s \in \mathbb{N}$, denote $R(s) = P_2(s) + \vec{h}$ and $R'(s) = P_{p-1}(s) - \vec{h}$. Since $P_2(s), P_{p-1}(s) \in \overline{X_1(s)X_p(s)} \subset \mathbb{Q}_{\geq}^p$ and \vec{h} is proportional to the vector $\overrightarrow{X_1(s)X_p(s)}$, the elements $R(s), R'(s)$ belong to r .

Proposition 9. *Let $S = \langle a_1, \dots, a_p \rangle$ be a numerical monoid. There exists $N_S \in \mathbb{N}$ such that $\overline{R(N_S)R'(N_S)} \subset \mathbb{Q}_{\geq}^p$ and such that for every $X \in \overline{R(N_S)R'(N_S)}$ and every $i \in \{2, \dots, p-1\}$ the element $X + (p-2)\vec{q}_i$ belongs to \mathbb{Q}_{\geq}^p .*

Proof. We have $R(s) = \frac{s(a_2-a_p)}{a_2(a_1-a_p)}e_1 + \frac{s(a_1-a_2)}{a_2(a_1-a_p)}e_p + \frac{d}{a_p-a_1}(a_pe_1 - a_1e_p)$ where $\frac{s(a_2-a_p)}{a_2(a_1-a_p)} > 0$ and $\frac{s(a_1-a_2)}{a_2(a_1-a_p)} > 0$ for every $s \in \mathbb{N} \setminus \{0\}$ (recall that $a_1 < \dots < a_p$). This implies that there exists \hat{s}_1 such that for all $s \geq \hat{s}_1$ we have $\frac{s(a_1-a_2)}{a_2(a_1-a_p)} > \frac{d}{a_p-a_1}a_1$, and thus $R(s) \in \mathbb{Q}_{\geq}^p$. Similarly, it can be easily proven that there exists \hat{s}_2 such that $R'(s) \in \mathbb{Q}_{\geq}^p$ for all $s \geq \hat{s}_2$. Since \mathbb{Q}_{\geq}^p is convex, it follows that $\overline{R(s)R'(s)} \subset \mathbb{Q}_{\geq}^p$ for all $s \geq \max\{\hat{s}_1, \hat{s}_2\}$.

For every s ,

$$\begin{aligned} R(s) + (p-2)\vec{q}_i &= P_2(s) + \vec{h} + (p-2)\vec{q}_i \\ &= \frac{s(a_2-a_p)}{a_2(a_1-a_p)}e_1 + \frac{s(a_1-a_2)}{a_2(a_1-a_p)}e_p + \frac{d}{a_p-a_1}(a_pe_1 - a_1e_p) \\ &\quad + \frac{(p-2)}{\gcd(a_i-a_p, a_p-a_1, a_1-a_i)}((a_i-a_p)e_1 + (a_p-a_1)e_i + (a_1-a_i)e_p) \\ &= \left(\frac{s(a_2-a_p)}{a_2(a_1-a_p)} + \frac{da_p}{a_p-a_1} + \frac{(p-2)(a_i-a_p)}{\gcd(a_i-a_p, a_p-a_1, a_1-a_i)} \right) e_1 \\ &\quad + \left(\frac{(p-2)(a_p-a_1)}{\gcd(a_i-a_p, a_p-a_1, a_1-a_i)} \right) e_i \\ &\quad + \left(\frac{s(a_1-a_2)}{a_2(a_1-a_p)} - \frac{da_1}{a_p-a_1} + \frac{(p-2)(a_1-a_i)}{\gcd(a_i-a_p, a_p-a_1, a_1-a_i)} \right) e_p. \end{aligned}$$

Since $a_1 < \dots < a_p$, the i th coordinate of $R(s) + (p-2)\vec{q}_i$ belongs to \mathbb{Q}_{\geq} . Furthermore, for every $s \in \mathbb{N} \setminus \{0\}$ we have $\frac{s(a_2-a_p)}{a_2(a_1-a_p)} > 0$ and $\frac{s(a_1-a_2)}{a_2(a_1-a_p)} > 0$. Hence, there exists $s' \in \mathbb{N}$ such that for every $s \geq s'$, $R(s) + (p-2)\vec{q}_i \in \mathbb{Q}_{\geq}^p$. Similarly, it must hold that there exists $s'' \in \mathbb{N}$ verifying that $R'(s) + (p-2)\vec{q}_i \in \mathbb{Q}_{\geq}^p$ for all $s \geq s''$. We take $N_S = \max\{s', s'', \hat{s}_1, \hat{s}_2\}$. Clearly, the elements $R(N_S)$ and $R'(N_S)$ are in \mathbb{Q}_{\geq}^p , and for all $i = 2, \dots, p-1$, $R(N_S) + (p-2)\vec{q}_i, R'(N_S) + (p-2)\vec{q}_i \in \mathbb{Q}_{\geq}^p$. Furthermore, for every $X \in \overline{R(N_S)R'(N_S)}$, the element $X + (p-2)\vec{q}_i$ belongs to the line segment with endpoints $R(N_S) + (p-2)\vec{q}_i \in \mathbb{Q}_{\geq}^p$ and $R'(N_S) + (p-2)\vec{q}_i \in \mathbb{Q}_{\geq}^p$. Using again that \mathbb{Q}_{\geq}^p is convex, we obtain that $X + (p-2)\vec{q}_i$ is also in \mathbb{Q}_{\geq}^p for all $i = 2, \dots, p-1$. \square

The numbers N_S fulfilling the conditions of the above proposition are not unique. We now define new elements that depend on the election of N_S . For the sake of simplicity in our notation, in the sequel, we will assume that for a given monoid S the natural number N_S represents an arbitrary fixed element of S fulfilling the conditions of Proposition 9.

Definition 10. *Let $S = \langle a_1, \dots, a_p \rangle$ be a numerical monoid. Define the vectors $\vec{w} = P_2(N_S) - X_1(N_S)$ and $\vec{w}' = P_{p-1}(N_S) - X_p(N_S)$. Note that $L(\vec{w}) = N_S/a_2 - N_S/a_1 < 0$ and $L(\vec{w}') = N_S/a_{p-1} - N_S/a_p > 0$.*

Lemma 11. *Let $S = \langle a_1, \dots, a_p \rangle$ be a numerical monoid and let $s \in \mathbb{N}$ be such that $s \geq N_S$. Then $L(X_2(s)) \leq L(X_1(s) + \vec{w})$ and $L(X_p(s) + \vec{w}') \leq L(X_{p-1}(s))$.*

Proof. Since $s \geq N_S$, we have $s(a_2 - a_1) \geq N_S(a_2 - a_1)$, and therefore $s(a_2 - a_1)/(a_1 a_2) \geq N_S(a_2 - a_1)/(a_1 a_2)$. Thus, $s/a_1 - s/a_2 \geq N_S/a_1 - N_S/a_2$, which implies that $L(X_1(s) + \vec{w}) = s/a_1 + N_S/a_2 - N_S/a_1 \geq s/a_2 = L(X_2(s))$. Similarly, we obtain the other inequality. \square

The following result generalizes Proposition 9.

Proposition 12. *Let $S = \langle a_1, \dots, a_p \rangle$ be a numerical monoid and let $s \in \mathbb{N}$ be such that $s \geq N_S$. Then, every element X of $(X_1(s) + \vec{w} + \vec{h})(X_p(s) + \vec{w}' - \vec{h})$ verifies $X \in \mathbb{Q}_{\geq}^p$ and $X + (p-2)\vec{q}_i \in \mathbb{Q}_{\geq}^p$ for all $i \in \{2, \dots, p-1\}$.*

Proof. The element $X_1(s) + \vec{w} + \vec{h}$ is equal to $\frac{s-N_S}{a_1}e_1 + R(N_S)$. By Proposition 9, the element $R(N_S)$ belongs to \mathbb{Q}_{\geq}^p . Thus, for every $s \geq N_S$ the element $X_1(s) + \vec{w} + \vec{h}$ is also in \mathbb{Q}_{\geq}^p . Similarly, it must hold $X_p(s) + \vec{w}' - \vec{h} \in \mathbb{Q}_{\geq}^p$ for every $s \geq N_S$. Thus, $(X_1(s) + \vec{w} + \vec{h})(X_p(s) + \vec{w}' - \vec{h}) \subset \mathbb{Q}_{\geq}^p$.

For every $i \in \{2, \dots, p-1\}$, the element $X_1(s) + \vec{w} + \vec{h} + (p-2)\vec{q}_i$ is equal to $\frac{s-N_S}{a_1}e_1 + R(N_S) + (p-2)\vec{q}_i$. By Proposition 9, $R(N_S) + (p-2)\vec{q}_i$ is in \mathbb{Q}_{\geq}^p . Thus for every $s \geq N_S$, we have $X_1(s) + \vec{w} + \vec{h} + (p-2)\vec{q}_i \in \mathbb{Q}_{\geq}^p$. Similarly, it can be obtained that $X_p(s) + \vec{w}' - \vec{h} + (p-2)\vec{q}_i \in \mathbb{Q}_{\geq}^p$. For every $X \in (X_1(s) + \vec{w} + \vec{h})(X_p(s) + \vec{w}' - \vec{h})$, the element $X + (p-2)\vec{q}_i$ belongs to the line segment with endpoints $(X_1(s) + \vec{w} + \vec{h}) + (p-2)\vec{q}_i \in \mathbb{Q}_{\geq}^p$ and $(X_p(s) + \vec{w}' - \vec{h}) + (p-2)\vec{q}_i \in \mathbb{Q}_{\geq}^p$. Hence, $X + (p-2)\vec{q}_i \in \mathbb{Q}_{\geq}^p$. \square

Corollary 13. *Let $s \in \mathbb{N}$ be such that $s \geq N_S$. For every $X \in (X_1(s) + \vec{w} + \vec{h})(X_p(s) + \vec{w}' - \vec{h})$, the set $\{X + \sum_{i=2}^{p-1} \lambda_i \vec{q}_i \mid 0 \leq \lambda_i \leq 1, \lambda_i \in \mathbb{Q}\}$ is contained in \mathbb{Q}_{\geq}^p .*

Proof. By Proposition 12, the elements $X, X + (p-2)\vec{q}_2, \dots, X + (p-2)\vec{q}_{p-1}$ are in \mathbb{Q}_{\geq}^p . The smallest convex set with respect the inclusion that contains the elements $X, X + (p-2)\vec{q}_2, \dots, X + (p-2)\vec{q}_{p-1}$ is the set $C = \{X + \sum_{i=2}^{p-1} \mu_i (p-2)\vec{q}_i \mid 0 \leq \sum_{i=2}^{p-1} \mu_i \leq 1, \mu_i \in \mathbb{Q}_{\geq}\}$. Since \mathbb{Q}_{\geq}^p is convex, the set C is a subset of \mathbb{Q}_{\geq}^p . The set C is equal to $\{X + \sum_{i=2}^{p-1} \lambda_i \vec{q}_i \mid 0 \leq \sum_{i=2}^{p-1} \lambda_i \leq p-2, \lambda_i \in \mathbb{Q}_{\geq}\}$ (just substitute $\mu_i(p-2)$ by λ_i), which clearly contains the set $\{X + \sum_{i=2}^{p-1} \lambda_i \vec{q}_i \mid 0 \leq \lambda_i \leq 1, \lambda_i \in \mathbb{Q}\}$. Thus, $\{X + \sum_{i=2}^{p-1} \lambda_i \vec{q}_i \mid 0 \leq \lambda_i \leq 1, \lambda_i \in \mathbb{Q}\} \subset \mathbb{Q}_{\geq}^p$. \square

In order to complete our construction, we express $Z(s)$ as a union of three different sets.

Definition 14. *Let $S = \langle a_1, \dots, a_p \rangle$ be a numerical monoid. For every $s \in \mathbb{N}$ such that $s \geq N_S$, define*

- $Z_1(s)$ the set of elements $x = (x_1, \dots, x_p) \in Z(s)$ verifying that $s/a_1 + L(\vec{w}) < L(x) \leq s/a_1$,

- $Z_2(s)$ the set of elements $x = (x_1, \dots, x_p) \in Z(s)$ verifying that $s/a_p + L(\vec{w}') - d \leq L(x) \leq s/a_1 + L(\vec{w}') + d$,
- $Z_3(s)$ the set of elements $x = (x_1, \dots, x_p) \in Z(s)$ verifying that $s/a_p \leq L(x) < s/a_p + L(\vec{w}')$.

For every $x = (x_1, \dots, x_p) \in Z(s)$ we have $s = x_1 a_1 + \dots + x_p a_p$. This implies $\frac{s}{a_1} = \frac{1}{a_1}(x_1 a_1 + \dots + x_p a_p)$, and using $a_1 < \dots < a_p$, we obtain $\frac{s}{a_1} = x_1 + x_2 \frac{a_2}{a_1} + \dots + x_p \frac{a_p}{a_1} \geq x_1 + \dots + x_p$. Similarly, it can be proved that $\frac{s}{a_p} \leq x_1 + \dots + x_p$. Since $L(x) = x_1 + \dots + x_p$, we obtain that $\frac{s}{a_1} = L(X_1(s)) \geq L(x) \geq L(X_p(s)) = \frac{s}{a_p}$, and hence $Z(s) = Z_1(s) \cup Z_2(s) \cup Z_3(s)$.

In the sequel, we use these sets to prove the periodicity of $\Delta(S)$, and to improve the algorithmic method that computes it.

Denote $\Delta(Z_i(s)) = \{l_{j+1} - l_j \mid j = 1, \dots, k-1\}$ with $\{l_1 < \dots < l_k\}$ the ordered set obtained from $\{L(x) \mid x \in Z_i(s)\}$ and by $\lfloor x \rfloor$ the largest integer not greater than x .

Theorem 15. *Let $s \in \mathbb{N}$ be such that $s \geq N_S$. Then $\Delta(Z_2(s)) = \{d\}$.*

Proof. Take $s \in \mathbb{N}$ such that $s \geq N_S$. Consider the set $K = \{X_p(s) + \vec{w}' - \vec{h}, X_p(s) + \vec{w}', X_p(s) + \vec{w}' + \vec{h}, \dots, X_p(s) + \vec{w}' + k\vec{h}, X_1(s) + \vec{w} + \vec{h}\}$ with $k \in \mathbb{N}$ the maximum such that $L(X_p(s) + \vec{w}' + k\vec{h}) \leq L(X_1(s) + \vec{w} + \vec{h})$. Denote by $\{l_0, l_1, \dots, l_{k+1}, l_{k+2}\}$ the set $L(K)$ where $l_i = L(X_p(s) + \vec{w}' + (i-1)\vec{h})$ with $i = 0, \dots, k+1$ and $l_{k+2} = L(X_1(s) + \vec{w} + \vec{h})$. We have $l_i - l_{i-1} = d$ (therefore $l_i = l_0 + id$) for every $i = 1, \dots, k+1$, and $l_{k+2} - l_{k+1} < d$. Since $\gcd(a_1, \dots, a_p) = 1$, there exists $\gamma = (\gamma_1, \dots, \gamma_p) \in \mathbb{Z}^p$ such that $\sum_{i=1}^p \gamma_i a_i = s$, and thus $\gamma \in \mathcal{H}_s$. Let \vec{v} be a minimal length increase vector. We can find $\xi \in \mathbb{Z}$ such that $l_0 \leq L(\gamma + \xi \vec{v}) = L(\gamma) + \xi d < l_0 + d = l_1$; since $\vec{v} \in M$ the element $\gamma + \xi \vec{v}$ is again in \mathcal{H}_s . From the linearity of L and the fact that $l_0 = L(X_p(s) + \vec{w}' - \vec{h}) \leq L(\gamma + \xi \vec{v}) < l_1 = L(X_p(s) + \vec{w}')$, we can assert that there exists $\gamma^0 \in (X_p(s) + \vec{w}' - \vec{h})(X_p(s) + \vec{w}') \subset \mathcal{H}_s$ such that $L(\gamma^0) = L(\gamma + \xi \vec{v})$ (just solve the linear equation $L(X_p(s) + \vec{w}' - \lambda \vec{h}) = L(\gamma + \xi \vec{v})$ on λ and take $\gamma^0 = X_p(s) + \vec{w}' - \lambda \vec{h}$). Since $\gamma + \xi \vec{v}, \gamma^0 \in \mathcal{H}_s$ and $L(\gamma + \xi \vec{v}) = L(\gamma^0)$, the element $\gamma + \xi \vec{v} - \gamma^0$ belongs to $\mathcal{H}_0 \cap V(E)$. The set $\{\vec{q}_2, \dots, \vec{q}_{p-1}\}$ is a basis of $\mathcal{H}_0 \cap V(E)$ (see Remark 6), this implies that there exist $\lambda_2, \dots, \lambda_{p-1} \in \mathbb{Q}$ such that $\gamma + \xi \vec{v} - \gamma^0 = \sum_{j=2}^{p-1} \lambda_j \vec{q}_j$. This leads to $\gamma + \xi \vec{v} - \sum_{j=2}^{p-1} \lfloor \lambda_j \rfloor \vec{q}_j = \gamma^0 + \sum_{j=2}^{p-1} (\lambda_j - \lfloor \lambda_j \rfloor) \vec{q}_j$. The element $\gamma + \xi \vec{v} - \sum_{j=2}^{p-1} \lfloor \lambda_j \rfloor \vec{q}_j$ belongs to \mathbb{Z}^p , and by Corollary 13 $\gamma^0 + \sum_{j=2}^{p-1} (\lambda_j - \lfloor \lambda_j \rfloor) \vec{q}_j \in \mathbb{Q}_{\geq}^p$. Thus, $\gamma^0 = \gamma + \xi \vec{v} - \sum_{j=2}^{p-1} \lfloor \lambda_j \rfloor \vec{q}_j$ belongs to \mathbb{N}^p and therefore $\gamma^0 \in Z(s)$. Using that $L(\vec{q}_i) = 0$, we obtain that γ^0 satisfies $l_0 \leq L(\gamma^0) = L(\gamma + \xi \vec{v} - \sum_{j=2}^{p-1} \lfloor \lambda_j \rfloor \vec{q}_j) = L(\gamma + \xi \vec{v}) < l_0 + d = l_1$. Now, for every $i = 1, \dots, k$, consider the elements $\gamma + \xi \vec{v} + i \vec{v}$; they verify that $l_i \leq L(\gamma + \xi \vec{v} + i \vec{v}) = L(\gamma + \xi \vec{v}) + id < l_{i+1}$. If we proceed similarly, we obtain $\gamma'^i \in Z(s)$ fulfilling that $l_i \leq L(\gamma'^i) = L(\gamma + \xi \vec{v}) + id < l_{i+1}$. In this way we get a sequence of elements $\gamma'^0, \dots, \gamma'^k$ with lengths equal to $L(\gamma'^0), L(\gamma'^0) + d, \dots, L(\gamma'^0) + kd$, respectively. If $l_{k+1} = l_{k+2}$, then $X_p(s) + \vec{w}' + k\vec{h} = X_1(s) + \vec{w} + \vec{h}$ and we only have to check if $L(\gamma'^0) + (k+1)d = l_{k+1}$. If so, then there exists γ'^{k+1} such that $L(\gamma'^{k+1}) = L(\gamma'^0) + (k+1)d$. If $l_{k+1} < l_{k+2}$, we check if $L(\gamma'^0) + (k+1)d \leq l_{k+2}$. If so, then repeating the procedure with the element

$\gamma + \xi \vec{v} + (k+1)\vec{v}$, we get an element γ'^{k+1} with $L(\gamma'^{k+1}) = L(\gamma'^0) + (k+1)d$. The sequence obtained is formed by k or $k+1$ elements $\gamma'^0, \dots, \gamma'^r$ whose ordered set of lengths is $\{L(\gamma'^0), L(\gamma'^0) + d, \dots, L(\gamma'^0) + rd\}$, $L(\gamma'^0) - l_0 < d$ and $l_{k+2} - (L(\gamma'^0) + rd) < d$. Since $d = \min(\Delta(S))$, it is not possible to find more elements having different lengths, and therefore $\Delta(Z_2(s)) = \{d\}$. \square

Corollary 16. *For all integer $s \geq N_S$, $Z_1(s) \cap Z_2(s) \neq \emptyset$ and $Z_2(s) \cap Z_3(s) \neq \emptyset$. Furthermore, $\Delta(s) = \Delta(Z_1(s)) \cup \{d\} \cup \Delta(Z_3(s))$.*

Proof. From Definition 14, $Z_1(s) \cap Z_2(s)$ is formed by the elements $x \in Z(s)$ verifying that $s/a_1 + L(\vec{w}) < L(x) \leq s/a_1 + L(\vec{w}) + d$. Taking into account that $d = (s/a_1 + L(\vec{w}) + d) - (s/a_1 + L(\vec{w}))$, and using the proof of Theorem 15, we can assert that there exists at least an element γ' such that $s/a_1 + L(\vec{w}) < L(\gamma') \leq s/a_1 + L(\vec{w}) + d$. This implies that $Z_1(s) \cap Z_2(s) \neq \emptyset$ and $\min\{L(x) \mid x \in Z_1(s)\}$ is equal to $\max\{L(x) \mid x \in Z_2(s)\}$. Similarly, it can be proved that $Z_2(s) \cap Z_3(s) \neq \emptyset$ and $\max\{L(x) \mid x \in Z_3(s)\}$ is equal to $\min\{L(x) \mid x \in Z_2(s)\}$. In this way, we obtain that $\Delta(s) = \Delta(Z_1(s)) \cup \Delta(Z_2(s)) \cup \Delta(Z_3(s))$. Since $s \geq N_S$, by Theorem 15, $\Delta(Z_2(s)) = \{d\}$, and thus $\Delta(s) = \Delta(Z_1(s)) \cup \{d\} \cup \Delta(Z_3(s))$. \square

The following result gives us the key to study the periodicity of $\Delta(S)$.

Theorem 17. *Let $S = \langle a_1, \dots, a_p \rangle$ be a numerical monoid and let $s \in \mathbb{N}$ be such that $s \geq N_S$. Then $Z_1(s + a_1) = \{x + e_1 \mid x \in Z_1(s)\}$ and $Z_3(s + a_p) = \{x + e_p \mid x \in Z_3(s)\}$.*

Proof. If $x \in Z_1(s)$, then $s/a_1 + L(\vec{w}) < L(x) \leq s/a_1$. Thus, $(s+a_1)/a_1 + L(\vec{w}) < L(x) + 1 = L(x + e_1) \leq (s+a_1)/a_1$, and therefore $x + e_1 \in Z_1(s + a_1)$.

Let $y = (y_1, \dots, y_p)$ be an element of $Z_1(s + a_1)$. Note that $(s + a_1)/a_1 + L(\vec{w}) < L(y) \leq (s + a_1)/a_1$, and thus $s/a_1 + L(\vec{w}) < L(y - e_1) \leq s/a_1$. If $y_1 > 0$, then $y - e_1 \in Z_1(s)$, and thus $y = (y - e_1) + e_1$ with $y - e_1 \in Z_1(s)$. Now assume $y_1 = 0$. The elements y and $((s + a_1)/a_2)e_2$ are both in \mathcal{H}_{s+a_1} . Thus, $y_2a_2 + \dots + y_pa_p = ((s + a_1)/a_2)a_2 = s + a_1$. Since $a_1 < \dots < a_p$ and $y \in \mathbb{Q}_{\geq}^p$, we obtain that $L(y) \leq (s + a_1)/a_2 = L(((s + a_1)/a_2)e_2)$. By Lemma 11, $L(X_2(s + a_1)) = (s + a_1)/a_2 \leq L(X_1(s + a_1) + \vec{w}) = (s + a_1)/a_1 + L(\vec{w})$. Thus $L(y) \leq (s + a_1)/a_2 \leq (s + a_1)/a_1 + L(\vec{w})$, contradicting the fact that $y \in Z_1(s + a_1)$.

Using the same argument, we also have $Z_3(s + a_p) = \{x + e_p \mid x \in Z_3(s)\}$. \square

Corollary 18. *Let $S = \langle a_1, \dots, a_p \rangle$ be a numerical monoid. Then $\Delta(S) = \bigcup_{s \in S, s \leq N_S + a_p - 1} \Delta(s)$. Furthermore, $\Delta(S)$ is periodic from N_S with period $\text{lcm}(a_1, a_p)$.*

Proof. From Theorem 17, it follows that $\Delta(Z_1(s + a_1)) = \Delta(Z_1(s))$ and $\Delta(Z_3(s + a_p)) = \Delta(Z_3(s))$ for all $s \geq N_S$. By Corollary 16, we obtain $\Delta(S) = \bigcup_{s=0}^{N_S + a_p - 1} \Delta(s)$.

Theorem 17 gives us that in the set $\{n \in \mathbb{N} \mid n \geq N_S\}$ the function $\Delta(Z_1(s))$ is periodic with period a_1 and $\Delta(Z_3(s))$ is periodic with period a_p . Hence, $\Delta(s)$ is periodic with period $\text{lcm}(a_1, a_p)$. \square

Remark 19. If $s \geq N_S$, then $d \in \Delta(s)$. Thus, $Z(s) \neq \emptyset$, and so the number $N_S - 1$ is greater than the Frobenius number of S .

3 Formulation of N_S

From Proposition 9, there exists $N_S \in \mathbb{N}$ fulfilling that $R(N_S)$, $R'(N_S)$, $P_2(N_S) + \vec{h} + (p-2)\vec{q}_i$ and $P_{p-1}(N_S) - \vec{h} + (p-2)\vec{q}_i$ belong to \mathbb{Q}_{\geq}^p . To compute N_S , we proceed as follows for every $i \in \{2, \dots, p-1\}$:

1. Consider the element $P_2(s) + \vec{h} + (p-2)\vec{q}_i$. This element is equal to

$$\begin{aligned} & \left(\frac{(p-2)(a_i - a_p)}{\gcd(a_i - a_1, a_1 - a_p, a_p - a_i)} + \frac{da_p}{a_p - a_1} + \frac{s(a_2 - a_p)}{a_2(a_1 - a_p)} \right) e_1 \\ & + \frac{(p-2)(a_p - a_1)}{\gcd(a_i - a_1, a_1 - a_p, a_p - a_i)} e_i \\ & + \left(\frac{(p-2)(a_1 - a_i)}{\gcd(a_i - a_1, a_1 - a_p, a_p - a_i)} + \frac{a_1 d}{a_1 - a_p} + \frac{(a_1 - a_2)s}{a_2(a_1 - a_p)} \right) e_p. \end{aligned}$$

Its i th coordinate is always positive and increasing the value of s we may obtain an element of \mathbb{Q}_{\geq}^p . Besides, if $P_2(s) + \vec{h} + (p-2)\vec{q}_i \in \mathbb{Q}_{\geq}^p$, then for every $s' \geq s$ the element $P_2(s') + \vec{h} + (p-2)\vec{q}_i$ is also in \mathbb{Q}_{\geq}^p . Note also that, since $((p-2)\vec{q}_i)_1 = \frac{(p-2)(a_i - a_p)}{\gcd(a_i - a_1, a_1 - a_p, a_p - a_i)} \leq 0$ and $((p-2)\vec{q}_i)_p = \frac{(p-2)(a_1 - a_i)}{\gcd(a_i - a_1, a_1 - a_p, a_p - a_i)} \leq 0$, if $P_2(s) + \vec{h} + (p-2)\vec{q}_i \in \mathbb{Q}_{\geq}^p$, then $R(s) = P_2(s) + \vec{h} \in \mathbb{Q}_{\geq}^p$.

2. Solve the linear equation $(P_2(s) + \vec{h} + (p-2)\vec{q}_i)_p = 0$ on s . Its solution is

$$S_i = -\frac{a_2(a_1 d \gcd(a_i - a_1, a_1 - a_p, a_p - a_i) + (p-2)(a_1 - a_i)(a_1 - a_p))}{(a_1 - a_2) \gcd(a_i - a_1, a_1 - a_p, a_p - a_i)}. \quad (1)$$

The element $P_2(S_i) + \vec{h} + (p-2)\vec{q}_i$ is equal to

$$\begin{aligned} & -\frac{a_2 d \gcd(a_i - a_1, a_1 - a_p, a_p - a_i) + (p-2)(a_2 - a_i)(a_1 - a_p)}{(a_1 - a_2) \gcd(a_i - a_1, a_1 - a_p, a_p - a_i)} e_1 \\ & + \frac{(p-2)(a_p - a_1)}{\gcd(a_i - a_1, a_1 - a_p, a_p - a_i)} e_i, \end{aligned}$$

it is an element of \mathbb{Q}_{\geq}^p , and therefore $R(s) \in \mathbb{Q}_{\geq}^p$ and $P_2(s) + \vec{h} + (p-2)\vec{q}_i \in \mathbb{Q}_{\geq}^p$ for all $s \geq S_i$, $s \in \mathbb{N}$.

3. We proceed similarly with the elements $P_{p-1}(s) - \vec{h} + (p-2)\vec{q}_i$ for every $i = 2, \dots, p-1$. In this case the solution of $(P_{p-1}(s) - \vec{h} + (p-2)\vec{q}_i)_1 = 0$ is

$$S'_i = \frac{a_{p-1}((p-2)(a_1 - a_p)(a_p - a_i) - da_p \gcd(a_i - a_1, a_1 - a_p, a_p - a_i))}{(a_{p-1} - a_p) \gcd(a_i - a_1, a_1 - a_p, a_p - a_i)}. \quad (2)$$

It is easy to check that $P_{p-1}(S'_i) - \vec{h} + (p-2)\vec{q}_i$ is again an element of \mathbb{Q}_{\geq}^p and $R'(s), P_{p-1}(s) - \vec{h} + (p-2)\vec{q}_i \in \mathbb{Q}_{\geq}^p$ for all $s \geq S'_i$, $s \in \mathbb{N}$.

4. We set N_S as the least integer greater than or equal to all the $2(p-2)$ solutions obtained:

$$N_S = \lceil \max(\{S_i \mid i = 2, \dots, p-1\} \cup \{S'_i \mid i = 2, \dots, p-1\}) \rceil, \quad (3)$$

where $\lceil x \rceil$ represents the smallest integer not less than x . This value fulfills the same properties than N_S in Proposition 9.

If we see our bound as a rational function on the variable a_p , the numerator has degree 2 and the denominator has degree 1. So, our bound increase linearly on a_p . On the contrary, the bound appearing in [5] is a polynomial of degree 2 on a_p . The results of Table 1 confirm the expected behaviors for both bounds.

Example 20. Let $S = \langle 15, 17, 27, 35 \rangle$. In this case the group M is generated by $\{(-13, 1, 4, 2), (-9, 0, 5, 0), (-7, 0, 0, 3)\}$. The lengths of these vectors are -6 , -4 and -4 . Since $\gcd(6, 4, 4) = 2$, the value of d is equal to 2. The solutions are the following

$$\begin{aligned} S_2 &= -\frac{a_2(a_1 d \gcd(a_2 - a_1, a_1 - a_4, a_4 - a_2) + (a_1 - a_2)(a_1 - a_4)(p-2))}{(a_1 - a_2) \gcd(a_2 - a_1, a_1 - a_4, a_4 - a_2)} = 595, \\ S_3 &= -\frac{a_2(a_1 d \gcd(a_3 - a_1, a_1 - a_4, a_4 - a_3) + (a_1 - a_3)(a_1 - a_4)(p-2))}{(a_1 - a_2) \gcd(a_3 - a_1, a_1 - a_4, a_4 - a_3)} = 1275, \\ S'_2 &= \frac{a_3((a_1 - a_4)(a_4 - a_2)(p-2) - a_4 d \gcd(a_2 - a_1, a_1 - a_4, a_4 - a_2))}{(a_3 - a_4) \gcd(a_2 - a_1, a_1 - a_4, a_4 - a_2)} = \frac{5805}{4} = 1451.25, \\ S'_3 &= \frac{a_3((a_1 - a_4)(a_4 - a_3)(p-2) - a_4 d \gcd(a_3 - a_1, a_1 - a_4, a_4 - a_3))}{(a_3 - a_4) \gcd(a_3 - a_1, a_1 - a_4, a_4 - a_3)} = \frac{2025}{4} = 506.25, \end{aligned}$$

and thus $N_S = \lceil \max(595, 1275, 1451.25, 506.25) \rceil = 1452$.

4 Computation of $\Delta(S)$

Lemma 2 and Formula 3 give us d and N_S , respectively. Both values are needed to compute $\Delta(S)$. The next step in our algorithm is the computation of $Z(N_S + a_p - 1), \dots, Z(N_S + a_p - 1 - a_1 - 1)$; each of these sets are the nonnegative integer solutions of a Diophantine equation. From these sets we define the sets

$$\Omega(s) = \{(x_1, L(x)) \mid x \in Z(s), x_1 = \max\{y_1 \mid y \in Z(s), L(y) = L(x)\}\}$$

for all $s = N_S + a_p - 1, \dots, N_S + a_p - 1 - a_1 - 1$. The second coordinate $L(x)$ is used in the computation of $\Delta(s)$; just project the second coordinate of the elements of $\Omega(s)$, order this set obtaining $\{l_1 < \dots < l_{t_s}\}$ and compute $\{l_{i+1} - l_i \mid i = 2, \dots, t_s\}$. It is straightforward to prove that for every $s \in \mathbb{N}$ the set $Z(s - a_1)$ is equal to $\{(x_1, \dots, x_p) - e_1 \mid (x_1, \dots, x_p) \in Z(s), x_1 \geq 1\}$, and therefore $\Omega(s - a_1) = \{(x_1 - 1, L(x) - 1) \mid (x_1, L(x)) \in \Omega(s), x_1 > 0\}$. This allows us to obtain all the sets $\Omega(s)$ with $s \in S$ and $s < N_S + a_p - 1 - a_1 - 1$ from the sets $\Omega(N_S + a_p - 1), \dots, \Omega(N_S + a_p - 1 - a_1 - 1)$.

Algorithm 21 collects the different improvements made in this work and computes the Delta set of a given numerical monoid. In addition, note that the computation of the sets $Z(*)$ and $\Omega(*)$ in steps 2 to 5 could be done in a parallel way. Our implementation of this algorithm in [9] takes into account this fact.

Algorithm 21. The input is $S = \langle a_1, \dots, a_p \rangle$ a numerical monoid. The output is the set $\Delta(S)$.

1. Using Lemma 2, compute d .

2. Using Equations (1), (2) and (3), compute N_S .
3. Compute the sets $Z(N_S + a_p - 1), \dots, Z(N_S + a_p - 1 - a_1 - 1)$.
4. Compute the sets $\Omega(N_S + a_p - 1), \dots, \Omega(N_S + a_p - 1 - a_1 - 1)$.
5. Compute $\{\Omega(s) \mid s \in S, s \leq N_S + a_p - 1\}$, using the sets of the preceding step.
6. Compute $\Upsilon = \{\Delta(s) \mid s \in S, s \leq N_S + a_p - 1\}$, using the sets of the preceding step.
7. Return $\Delta(S) = \cup_{\Psi \in \Upsilon} \Psi$.

Solve a problem by using an exhaustive search usually have two computational disadvantages: the common high size of the bounds and the complexity to check the corresponding properties; but these computational troubles have to be assumed if there exists no other way to solve the problem. The algorithm method introduced in this work and the method that appears in [5] are based on computing the set of factorizations of some bounded elements belonging to a numerical monoid. The size of our bound $N_S + a_p - 1$ (equation (3)) seems to be better than the bound of [5, Theorem 1].

Another remarkable fact is that for each element in the set $\{N_S + a_p - 1, \dots, N_S + a_p - 1 - a_1 - 1\}$, it is necessary to compute its set of factorizations by solving its defining equation. This is a high computational complexity problem (NP-complete problem) although there exist some specific algorithms for solving a unique Diophantine equation (see [8]). Anyway, since our bound reduces the size of the checking region, Algorithm 21 needs to do less high complexity computations than the algorithm obtained from [5, Corollary 3]. So the usual computational disadvantages are partly avoided.

The above computational considerations are reflected in the following examples.

Example 22. We compare our bound with the bound of [5]. Using the functions `ChapmanBound` and `Bound` of [9] we obtain Table 1. For the semigroup $S = \langle 15, 16, 27 \rangle$ these functions can be used as follows:

```
In[1] := Bound[{15, 16, 27}]
Out[1] = 446
...
In[2] := ChapmanBound[{15, 16, 27}]
Out[2] = 70224
```

Table 1: Comparison with bound of [5]

Semigroup	Bound of [5]	$N_S + a_p - 1$
$\langle 15, 16, 27 \rangle$	70224	446
$\langle 37, 59, 101 \rangle$	3613337	2208
$\langle 201, 451, 577 \rangle$	900996525	89119
$\langle 15, 17, 27, 35 \rangle$	166855	1466
$\langle 100, 121, 142, 163, 284 \rangle$	97605860	201052
$\langle 1001, 1211, 1421, 1631, 2841 \rangle$	97744425121	2064141

Clearly, our bound is lower than the bound of Theorem 1 of [5].

Example 23. To compute the Delta set of a numerical monoid can be used the function `DeltaSNParallel` (see [9]) as follows:

```
In[3]:= AbsoluteTiming [DeltaSNParallel[{4, 6, 15}]]
Out[3]={0.018001, {1, 2, 3}}
```

The returned value is $\{0.018001, \{1, 2, 3\}\}$ where 0.018001 is the time required for its computation, and $\{1, 2, 3\}$ is the Delta set of the numerical monoid $S = \langle 4, 6, 15 \rangle$. Table 2 contains the computation of $\Delta(S)$ for some numerical monoids. All these examples have been done in an Intel Core i7 with 16 GB of main memory using the parallel version of the program.

Table 2: Computation of $\Delta(S)$

Semigroup	Bound of [5]	$N_S + a_p - 1$	$\Delta(S)$	Time (seconds)
$\langle 4, 6, 15 \rangle$	8124	81	$\{1, 2, 3\}$	0.018
$\langle 4, 6, 199 \rangle$	1425660	1185	$\{1, 2, \dots, 65\}$	0.152
$\langle 11, 37, 52, 93 \rangle$	2560511	5952	$\{1, \dots, 21\}$	5.205
$\langle 51, 53, 55, 117 \rangle$	5806839	9749	$\{2, 4, 6\}$	6.962
$\langle 11, 53, 73, 87 \rangle$	3209839	14391	$\{2, 4, 6, 8, 10, 22\}$	33.098
$\langle 11, 53, 73, 81 \rangle$	2782447	6599	$\{2, 4, 6, 8, 10, 12\}$	3.713
$\langle 7, 15, 17, 18, 20 \rangle$	60105	1941	$\{1, 2, 3\}$	151.668
$\langle 10, 17, 19, 25, 31 \rangle$	163540	1189	$\{1, 2, 3\}$	8.629
$\langle 10, 17, 19, 21, 25 \rangle$	106420	2031	$\{1, 2\}$	102.656
$\langle 7, 19, 20, 25, 29 \rangle$	159923	3900	$\{1, 2, 3, 5\}$	878.702
$\langle 31, 73, 77, 87, 91 \rangle$	6047393	31394	$\{2, 4, 6\}$	24012.245

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